Naively, it may seem that our everyday friend electromagnetism (responsible for circuits, light, atomic and molecular bonds) bears little if any resemblance to the less obvious strong and weak interactions. However when properly understood as gauge theories (with local invariance) we not only see how these three forces are similar, but the key feature that makes their roles in our universe so different ... whether or not the relevant group is abelian.

Recap: E&M

$$\mathcal{L} = \underbrace{\hbar c \bar{\psi} \gamma^\mu \partial_\mu \psi + i q \hbar c \bar{\psi} \gamma^\mu A_\mu \psi}_{\hbar c \bar{\psi} \gamma^\mu D_\mu \psi} + mc^2 \bar{\psi} \psi + \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$$

w/ $D_\mu \equiv \partial_\mu + i q A_\mu$

$$F_{\mu\nu} = -\frac{i}{q} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Invariant under local U(1):

$\mathcal{L} \to \mathcal{L}' = \mathcal{L}$ under $\psi \to \psi' = e^{iq\phi(x^\mu)} \psi$, $\bar{\psi} \to \bar{\psi}' = \bar{\psi} e^{-iq\phi(x^\mu)}$, $A_\mu \to A_\mu' = A_\mu - \partial_\mu \phi(x^\mu)$

QCD

The strong interactions are understood as a theory of local SU(3) invariance acting on quarks only.

Start with free Dirac (since quarks, like all matter, are spin $\frac{1}{2}$). Since we need something for the SU(3) matrices to act on, we consider a 3-component quark field $\psi = \underbrace{\begin{pmatrix} \psi_r \\ \psi_b \\ \psi_g \end{pmatrix}}$ where this would correspond to three versions of the same particular quark, e.g. a red up, blue up, g up. I hope it is clear that this has nothing to do with our usual notion of "color". Here it is just a convenient way of labelling something with 3 options. Note, r, b, g are not 3 values of the same charge (like $\pm e$ in E&M), but rather they are 3 distinct types of charge, each with 2 values $r, \bar{r}, b, \bar{b}, g, \bar{g}$.

r, b, g are the basis states in "color" space.

$$\mathcal{L} = \hbar c \bar{\psi} \gamma^\mu \partial_\mu \psi + mc^2 \bar{\psi} \psi \qquad \bar{\psi} = i \psi^+ \gamma^0$$ (This includes a transpose in color space)

As writ, this is invariant under global $\psi \to \psi' = e^{-i \frac{q}{\hbar c} \lambda \cdot \phi} \psi$

Recall the exponential map from Lie algebras to Lie group elements: $A = e^{i g_A v^A}$

g_A is a vector of matrix generators
v^A is a vector of parameters

Comparing these: $\lambda = g_A$ 8 generators of SU(3)

$\phi = v^A$ 8 parameters (one for each generator)

$-\frac{q}{\hbar c} \equiv -g$ = some constant which will eventually determine the coupling strength

Note, even though there are 3 types of charge, there is only one coupling. This is because SU(3) transformations "mix-up" the charges, and symmetry would then require they all couple equally.

Note: I am writing this with an $\hbar c$ and a $(-)$. This is all convention, but will parallel Griffith's treatment.

$$\mathcal{L} = \hbar c \bar{\psi}\gamma^\mu \partial_\mu \psi + mc^2 \bar{\psi}\psi \qquad \bar{\psi} = \psi^\dagger \gamma^0 \qquad \psi = \begin{pmatrix} \psi_r \\ \psi_b \\ \psi_g \end{pmatrix}$$

We now want to make this invariant under local SU(3): $\psi \to \psi' = e^{-ig\lambda \cdot \phi(x^\mu)}\psi$

Following the example from U(1) as far as we can.

Let $\partial_\mu \Rightarrow D_\mu = \partial_\mu + ig \underbrace{\lambda \cdot A_\mu}$ (from U(1) we had $D_\mu = \partial_\mu + iqA_\mu$)

We need 8 gauge fields, one for each generator.

If it isn't obvious that we would need 8 gauge fields, recall that even though a general SU(3) element is $e^{ig_A v^A}$, we could do a transformation using only one generator, e.g. $v^A = (1,0,0,\cdots)$. Then we could go through the process of making this local. Then do the same w/ $v^A = (0,1,0,0\cdots)$. In all we would do this 8 times, getting 8 gauge fields that pair up w/ each generator.

We want the new derivative to be "covariant", i.e. $D_\mu\psi \to D'_\mu\psi' = e^{-ig\lambda\cdot\phi(x^\mu)} D_\mu\psi$
(that way the overall transformation cancels the one from $\bar{\psi}$.)

For this to happen: $D_\mu\psi = \partial_\mu\psi + ig\lambda\cdot A_\mu\psi \to \partial_\mu\psi' + ig\lambda\cdot A'_\mu\psi'$

$$= \partial_\mu(e^{-ig\lambda\cdot\phi}\psi) + ig\lambda\cdot A'_\mu e^{-ig\lambda\cdot\phi}\psi$$

$$= \partial_\mu(e^{-ig\lambda\cdot\phi})\psi + e^{-ig\lambda\cdot\phi}\partial_\mu\psi + ig\lambda\cdot A'_\mu e^{-ig\lambda\cdot\phi}\psi$$

$$= e^{-ig\lambda\cdot\phi}(\partial_\mu\psi + ig\lambda\cdot A_\mu\psi)$$

$$\text{if} \quad \lambda\cdot A'_\mu = e^{-ig\lambda\cdot\phi}\lambda\cdot A_\mu e^{ig\lambda\cdot\phi} + \frac{i}{g}\partial_\mu(e^{-ig\lambda\cdot\phi})e^{ig\lambda\cdot\phi}$$

Compare this to U(1) where there was only one generator so $\lambda\cdot A_\mu \sim A_\mu \Rightarrow A'_\mu = A_\mu - \partial_\mu\phi$. We can see why U(1) was easier. In that case λ was a number so everything commuted and we could move the exponentials around to let them cancel. For SU(3), λ is a matrix, so nothing can be freely moved around, thus we need to leave expressions like this as they are.

As before we notice that we have introduced interactions between ψ and A_μ, $\mathcal{L}_{int} = ig\hbar c\bar{\psi}\gamma^\mu\lambda\cdot A_\mu\psi$. These describe quark fields ψ interacting w/ gluons A_μ.

To complete the story, we need to give the new gauge fields A_μ their own kinetic term.

For $U(1)$ we used: $\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$ w/ $F_{\mu\nu} = -\frac{i}{q}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$
and we observed that this is the Proca Lagrangian w/ $m=0$ (as needed for gauge invariance).

For QCD we can again use $\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$ (Proca w/ $m=0$) but this time:

$$-\frac{i}{g}[D_\mu, D_\nu]\psi = -\frac{i}{g}(\partial_\mu + ig\lambda\cdot A_\mu)(\partial_\nu\psi + ig\lambda\cdot A_\nu\psi) + \frac{i}{g}(\partial_\nu + ig\lambda\cdot A_\nu)(\partial_\mu\psi + ig\lambda\cdot A_\mu\psi)$$

$$= -\frac{i}{g}\partial_\mu\partial_\nu\psi + \frac{i}{g}\partial_\nu\partial_\mu\psi + \lambda\cdot A_\nu\partial_\mu\psi + \lambda\cdot A_\mu\partial_\nu\psi - \lambda\cdot A_\nu\partial_\mu\psi - \lambda\cdot A_\mu\partial_\nu\psi$$

$$+ \partial_\mu(\lambda\cdot A_\nu)\psi - \partial_\nu(\lambda\cdot A_\mu)\psi$$

$$+ ig(\lambda\cdot A_\mu)(\lambda\cdot A_\nu)\psi - ig(\lambda\cdot A_\nu)(\lambda\cdot A_\mu)\psi$$

So $F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu] = \partial_\mu(\lambda\cdot A_\nu) - \partial_\nu(\lambda\cdot A_\mu) + ig[\lambda\cdot A_\mu, \lambda\cdot A_\nu]$

If we denote $\lambda\cdot A_\mu = \lambda^a A_\mu^a$ or $\lambda\cdot A_\mu = \lambda^b A_\mu^b$, etc. w/ $a,b,c = 1,2,\cdots,8$

Then: $F_{\mu\nu} = \lambda^a(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) + ig[\lambda^b, \lambda^c] A_\mu^b A_\nu^c$

Recall that for the generators of $SU(3)$ [HW3 problem 4]: $[g_i, g_j] = i\underbrace{f^{ijk}}_{\text{Structure constants of } SU(3)} g_k$

$$\Downarrow$$

$$[\lambda^b, \lambda^c] = i f^{abc}\lambda^a$$

So: $F_{\mu\nu} = \lambda^a(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) - g f^{abc}\lambda^a A_\mu^b A_\nu^c$

Or we can think of this as $F_{\mu\nu} = \lambda^a F_{\mu\nu}^a$ w/ $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c$

Now that we have the field strength $F_{\mu\nu}$, we add the gauge invariant term:

$$\mathcal{L}_{A_\mu} = \frac{1}{16\pi} F_{\mu\nu}^{\ a} F^{\mu\nu a} = \frac{1}{16\pi}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} - g f^{ade} A^{\mu d} A^{\nu e}\right)$$

$$= \underbrace{\frac{1}{16\pi}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}\right)}_{\text{usual kinetic term}} \left.\begin{aligned} &- \frac{g}{16\pi} f^{ade} A_\mu^d A_\nu^e \left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}\right) \\ &- \frac{g}{16\pi} f^{abc} A^{\mu b} A^{\nu c}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) \\ &+ \frac{g^2}{16\pi} f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} \end{aligned}\right\} \begin{array}{l}\text{These are}\\ \text{gluon-gluon}\\ \text{interactions!}\end{array}$$

Note that the gluon-gluon interactions critically depend on SU(3) being non-abelian, i.e. $f^{abc} \neq 0$. This is of course why photons in (abelian U(1)) E&M do not interact w/ each other (at least classically).

These gluon-gluon interactions bring in a host of new effects including glueballs, confinement, etc.

A useful way to think about quark gluon interactions is as follows.

Consider $\psi = \begin{pmatrix} \psi_r \\ \psi_b \\ \psi_g \end{pmatrix}$ and A'_μ associated w/ $\lambda' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ [again from HW3]

Then: $\bar\psi \lambda \cdot A'_\mu \psi = (\bar\psi_r \; \bar\psi_b \; \bar\psi_g) \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\frac{1}{\sqrt{2}}(\underbrace{b\bar{r}}_{b\bar{r}(r)=b} + \underbrace{r\bar{b}}_{r\bar{b}(b)=r})} \begin{pmatrix} \psi_r \\ \psi_b \\ \psi_g \end{pmatrix} = \bar\psi_r \psi_b + \bar\psi_b \psi_r + \bar\psi_g \psi_g$

So the gluons are bi-colored ($c\bar{c}$) while the quarks are just colored (r, b, g), anti-quarks $(\bar{r}, \bar{b}, \bar{g})$.
This will be immensely helpful in constructing Feynman diagrams for QCD.